Global dynamics in Hopfield's model

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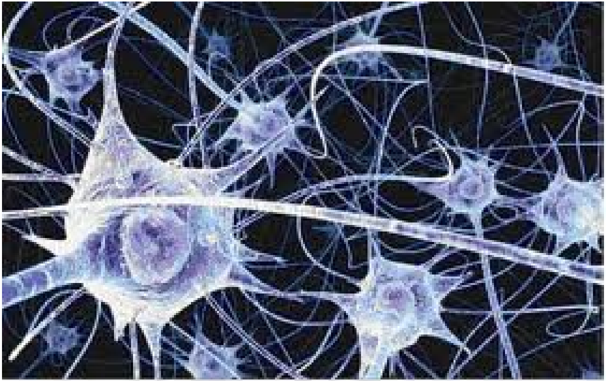
This is a joint work with E. Liz

This talk is based on these two papers:

- E. Liz and A. R-H., Global dynamics in discrete neural networks allowing nonmonotonic feedback.
- E. Liz and A. R.-H, Attractivity, Multistability, and Global bifurcation in Hopfield's model



DESCRIPTION OF THE MODEL

Consider a network of s-neurons. The variable $x_i(t)$ represents the voltage state of the neuron i at time t. This state can be modified by internal processes (inside the network) and external processes.

Key Assumption: Input from other neurons and stimuli are additive.

$$x_i'(t) = \underbrace{\mathit{Internal\ processes}} + \overbrace{\mathit{External\ inputs}} \tag{1}$$

External processes: We denote by $E_i(t)$ the external input in neuron i.

Internal processes, (inside the network):

- Internal decay inside the neuron (α represents the lost voltage inside the neuron)
- Connection between neurons. The key elements are:
  - **Signal activation function**. The way of transmission between the neurons. Concrete examples, $f(x) = \tanh(x)$, or the nonmonotone Morita's activation function

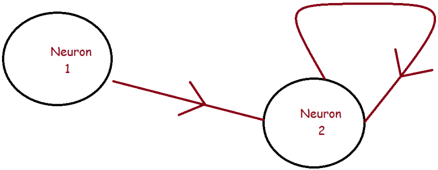
$$f(x) = \frac{1 - \exp(-\alpha x)}{1 + \exp(-\alpha x)} \times \frac{1 + k \exp(\beta(|x| - h))}{1 + \exp(\beta(|x| - h))}, \qquad (2)$$

  - **Connection matrix** $T = (w_{ij})$ represents the connections strengths between neurons. Assume that there is a connection from neuron j to i. If the output excitates (resp. Inhibits) neuron i then $w_{ij} > 0$ (resp. $w_{ij} < 0$.)
  - **Time delays**: We introduce time delays due to the finite velocity of propagation.

Then our model can be written as

$$x_i'(t) = \overbrace{-\alpha x_i(t) + \sum_{j=1}^{s} w_{ij} f(x_j(t - \tau_{ij}))}^{\textit{Internal processes}} + \underbrace{E_i(t)}_{\textit{External processes}} \qquad (3)$$

Example 1:

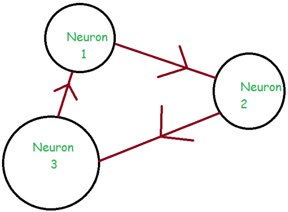


We have an autoconnection in neuron 2.
We have a connection from neuron 1 to neuron 2.
We have no external inputs.

In this situation our model can be written in this way:

$$\begin{cases} x_1'(t) = -\alpha x_1(t) \\ x_2'(t) = -\alpha x_2(t) + w_{21} f(x_1(t-\tau_{21})) + w_{22} f(x_2(t-\tau_{22})) \end{cases}$$

Example 2:



We have a connection from neuron 1 to neuron 2.
We have a connection from neuron 2 to neuron 3.
We have a connection from neuron 3 to neuron 1.

In this situation, our model can be written in this way

$$\begin{cases} x_1'(t) = -\alpha x_1(t) + w_{13} f(x_3(t-\tau_{13})) \\ x_2'(t) = -\alpha x_2(t) + w_{21} f(x_1(t-\tau_{21})) \\ x_3'(t) = -\alpha x_3(t) + w_{32} f(x_2(t-\tau_{32})) \end{cases}$$

This model was originally proposed by Hopfield in

- J. Hopfield, Neural networks and physical systems with emergent collective computational abilities, *Proc. Natl. Acad. Sci.* **79** (1982), 2554–2558

and later modified by Marcus and Westervelt introducing time delays

- C. M. Marcus and R. M. Westervelt, Stability of analog neural networks with delay, *Phys. Rev. A* **39** (1989), 347–359.

The previous system has been applied in different areas such as classification, associative memory, pattern recognition, parallel computations, and optimization

Our aim is to study some dynamical behaviors of

$$x_i'(t) = -x_i(t) + \sum_{j=1}^{s} w_{ij} f(x_j(t - \tau_{ij})) + E_i(t) \tag{4}$$

The natural phase space for (4) is $X = \mathcal{C}([-\tau, 0], \mathbb{R}^s)$, equipped with the max-norm

$$|\phi|_\infty = \max\{|\phi(t)| : t \in [-\tau, 0]\}.$$

For each $\phi \in X$, we employ the notation

$$x(t, \phi) = (x_1(t, \phi), x_2(t, \phi), \ldots, x_s(t, \phi))$$

to refer to the solution of (4) with initial condition ϕ.

Abstract framework: Consider the system of delay differential equations

$$x_i'(t) = -x_i(t) + F_i(x_1(t-\tau_{i1}), x_2(t-\tau_{i2}), \ldots, x_s(t-\tau_{is})), \qquad 1 \le i \le s \tag{5}$$

where $F_i : \mathbb{R}^s \longrightarrow \mathbb{R}$ is locally Lipschitz-continuous and $\tau_{ij} \ge 0$. Our goal is to deduce some properties of stability in the previous via the discrete system

$$x(N+1) = F(x(N)), \;\; N = 0, 1, \ldots, \tag{6}$$

where $x(N) = (x_1(N), \ldots, x_s(N)) \in \mathbb{R}^s$ and $F = (F_1, \ldots, F_s)$.

VERY IMPORTANT: WE WILL STUDY A DYNAMICAL SYSTEM IN AN INFINITE DIMENSIONAL SPACE FROM A CONCRETE DYNAMICAL SYSTEM IN FINITE DIMENSION!!!

Definition

Consider $F : D \subset \mathbb{R}^s \longrightarrow D$ a continuous map defined on $D = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_s, b_s)$. An equilibrium $z_* \in D$ of the system

$$x(N+1) = F(x(N)),\ N = 0, 1, \ldots, \tag{7}$$

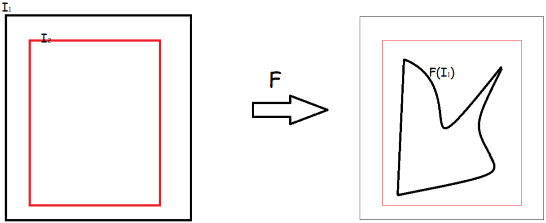
is a strong attractor in D if for every compact set $K \subset D$ there exists a family of sets $\{I_n\}_{n\in\mathbb{N}}$, where I_n is the product of s nonempty compact intervals, satisfying that

(B1) $z_* \in Int(I_n)$ for all $n \in \mathbb{N}$.

(B2) $K \subset Int(I_1) \subset D$.

(B3) $F(I_n) \subset I_{n+1} \subset Int(I_n)$ and $\displaystyle\bigcap_{n=1}^{\infty} I_n = \{z_*\}$.

Illustration of the notion of strong attractor



Observe that, in one dimension, the definitions of attraction and strong attraction coincide. However, this result is not true in higher dimensions.

$$x_i'(t) = -x_i(t) + F_i(x_1(t-\tau_{i1}), x_2(t-\tau_{i2}), \dots, x_s(t-\tau_{is})), \qquad 1 \le i \le s \tag{8}$$

Theorem

Assume that $F : D \subset \mathbb{R}^s \longrightarrow D$ *is a continuous map defined on* $D = (a_1, b_1) \times \cdots \times (a_s, b_s)$ *and* $z_* \in \mathbb{R}^s$ *is a strong attractor for*

$$x(N+1) = F(x(N)), \quad N = 0, 1, \dots \tag{9}$$

in D*. Then, for each* $\phi \in \{x \in \mathcal{C}([-\tau, 0], \mathbb{R}^s) : x(t) \in D \text{ for all } t \in [-\tau, 0]\}$, $\lim_{t\to\infty} x(t, \phi) = z_*$.

The notion of strong attractor is essential in the previous theorem.

The notion of strong attractor is essential in the previous theorem.Consider

$$F(x,y) = \begin{cases} (x(1+x),0) & \text{if} \quad y \geq 1 \\ (x(1+x), y(1-y)) & \text{if} \quad \frac{1}{2} \leq y \leq 1, \\ (x(1+x), y - \frac{1}{4}) & \text{if} \quad \frac{1}{4} \leq y \leq \frac{1}{2} \\ (4yx(1+x),0) & \text{if} \quad y \leq \frac{1}{4} \end{cases}$$

It is easy to prove that $F^3(x,y) = (0,0)$ and so $(0,0)$ is an attractor for the discrete system associated with F.

Consider

$$\begin{cases} x'(t) = -x(t) + F_1(x(t), y(t)) \\ y'(t) = -y(t) + F_2(x(t), y(t)) \end{cases} \tag{10}$$

When $\frac{1}{2} \leq y \leq 1$

$$\begin{cases} x'(t) = x^2(t) \\ y'(t) = -y(t)^2 \end{cases} \tag{11}$$

where $(x(t), y(t)) = (\frac{-1}{t-2}, \frac{1}{t}) \in \mathbb{R} \times (\frac{1}{2}, 1)$ defined on $1 < t < 2$ is an unbounded solution of our system. It is clear that this solution does not tend to zero.

At this moment, it is possible to apply our strategy when our system has an expression of the type

$$x_i'(t) = -x_i(t) + F_i(x_1(t-\tau_{i1}), x_2(t-\tau_{i2}), \ldots, x_s(t-\tau_{is})), \qquad 1 \leq i \leq s. \tag{12}$$

Our next aim will be to extend our theorem in a general framework.

Theorem

The previous theorem can be adapted to consider systems of the type

$$x_i'(t) = -f_i(x_i(t)) + F_i(x_1(t-\tau_{i1}), x_2(t-\tau_{i2}), \dots, x_s(t-\tau_{is})), \quad (13)$$

provided $f_i : \mathbb{R} \longrightarrow \mathbb{R}$ *is a homeomorphism for every* $i = 1, 2, \dots, s$. *In this case we have to assume that* z^* *is a strong attractor for the discrete system*

$$x(N+1) = G(x(N)), \quad N = 0, 1, 2, \dots$$

where
$G(x_1, \dots, x_s) = (f_1^{-1}(F_1(x_1, \dots, x_s)), \dots, f_s^{-1}(F_s(x_1, \dots, x_s)))$.

Models with distributed delay.

We study the following system:

$$u_i'(t) = -u_i(t) + \sum_{j=1}^{s} w_{ij} \int_{-\infty}^{0} \theta_{ij}(s) f(u_j(t+s))ds, \quad 1 \le i \le s, \tag{14}$$

where $\theta_{ij} : (-\infty, 0] \longrightarrow [0, +\infty)$ are positive integrable functions satisfying that $\theta_{ij}(s) = 0$ when $s \le \tau_{ij} \in (-\infty, 0)$, and $\int_{-\infty}^{0} \theta_{ij}(s)ds = 1$.

We can get some properties of global stability for system (14) via the discrete system

$$F(x_1, \dots, x_s) = \left(\sum_{j=1}^{s} w_{1j} f(x_j), \dots, \sum_{j=1}^{s} w_{sj} f(x_j) \right).$$

Theorem

Assume that f *is continuous and* $z_* \in \mathbb{R}^s$ *is a strong attractor for*

$$x(N+1) = F(x(N)), \quad N = 0, 1, \dots \tag{15}$$

on $D = (a_1, b_1) \times \cdots \times (a_s, b_s)$.
Then, for each
$\phi \in \{x \in \mathcal{C}([-\tau, 0], \mathbb{R}^s) : x(t) \in D \text{ for all } t \in [-\tau, 0]\}$, *the solution* $u(t, \phi)$ *with initial condition* ϕ *satisfies that* $\lim_{t \to \infty} u(t, \phi) = z_*$.

It is also important to recall that the connection of some dynamical behaviors of a discrete equation with some properties of a scalar delay differential equation (DDE) is not new; a systematic approach was initiated by early papers of Mallet-Paret and Nussbaum

- J.Mallet-Paret, R.Nussbaum, A differential-delay equation arising in optics and physiology, SIAM J. Math. Anal., 20 (1989), 249–292.

Now we apply the previous results to

$$x_i'(t) = -x_i(t) + \sum_{j=1}^{s} w_{ij} f(x_j(t - \tau_{ij})),\ 1 \le i \le s. \qquad (16)$$

Next we list some properties that we will assume for the activation functions:

(Ac1) f is of class $\mathcal{C}^1$,

(Ac2) $f(0) = 0$ with $f'(0) > 0$,

(Ac3) f is odd (i.e. $f(x) = -f(-x)$) and bounded.

The connection matrix is normalized as

$$\sum_{j=1}^{s} |w_{ij}| = 1 \text{ for } \ 1 \le i \le s. \qquad (17)$$

Now we deduce some properties of global attraction in our system of delay differential equations from the dynamics of

$$x(N+1) = f(x(N)) \tag{18}$$

Attractivity of the trivial solution

Consider

$$x_i'(t) = -x_i(t) + \sum_{j=1}^{s} w_{ij} f(x_j(t - \tau_{ij})), \; 1 \leq i \leq s. \qquad (19)$$

Theorem

Assume that 0 is an attractor for

$$y(N+1) = f(y(N)) \qquad (20)$$

in $(-\Delta, \Delta)$. Then every solution of our system with initial condition in

$$\{\phi \in \mathcal{C}([-\tau, 0], \mathbb{R}^s) : \phi(t) \in (-\Delta, \Delta)^s \;\; \textit{for all} \;\; -\tau \leq t \leq 0\}$$

satisfies that $\lim_{t\to\infty} x(t, \phi) = (0, \ldots, 0)$.

Proof.

Consider a map $H : D \subset \mathbb{R}^s \longrightarrow \mathbb{R}^s$ defined by

$$H(x_1, \dots, x_s) = \left(\sum_{j=1}^{s} w_{1j} f(x_j), \dots, \sum_{j=1}^{s} w_{sj} f(x_j) \right),$$

where $w_{ij} \in \mathbb{R}$ for all $i, j \in \{1, \dots, s\}$ with $\sum_{j=1}^{s} |w_{ij}| = 1$. We prove the following result: Assume that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and odd (i.e. $f(x) = -f(-x)$ for all $x \in \mathbb{R}$). If there is $\Delta \in (0, \infty]$ such that $0 \in \mathbb{R}$ is an attractor for the one-dimensional discrete system

$$x(N+1) = f(x(N)), \ N = 0, 1, \dots, \tag{21}$$

in $(-\Delta, \Delta)$ then $0 \in \mathbb{R}^s$ is a strong attractor for the dynamical system associated with H in $(-\Delta, \Delta)^s$.

Attractivity of non-trivial equilibria

Consider

$$x_i'(t) = -x_i(t) + \sum_{j=1}^{s} w_{ij} f(x_j(t - \tau_{ij})),\ 1 \le i \le s. \tag{22}$$

Theorem

Assume excitatory connections in the previous system, that is $w_{ij} \ge 0$ for every index and that x_+ is an attractor for

$$y(N+1) = f(y(N)) \tag{23}$$

in $(0, \Delta)$. Then every solution of our system with initial condition in

$$\{\phi \in \mathcal{C}([-\tau, 0], \mathbb{R}^s) : \phi(t) \in ((0, \Delta)^s \ \textit{for all}\ -\tau \le t \le 0\}$$

satisfies that $\lim_{t\to\infty} x(t, \phi) = (x_+, \ldots, x_+)$.

Proof

Now we consider the map $G : \mathbb{R}^s \longrightarrow \mathbb{R}^s$ defined by

$$G(x_1, \dots, x_s) = \left(\sum_{j=1}^{s} w_{1j} f(x_j), \dots, \sum_{j=1}^{s} w_{sj} f(x_j) \right),$$

where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and satisfies the following assumptions:

(C1) $f(0) = 0$,

(C2) $f((0, \beta)) \subset (0, \beta)$ with $\beta \in (0, +\infty]$,

(C3) there exists $x_* \in (0, \beta)$ such that $f(x_*) = x_*$.

Assume that all coefficients w_{ij} are nonnegative and

$$\sum_{j=1}^{s} w_{ij} = 1,$$

for all $i \in \{1, \ldots, s\}$. If $x_* > 0$ is an attractor for

$$x(N+1) = f(x(N)), \ N = 0, 1, \ldots, \tag{24}$$

in $(0, \beta)$, then $(x_*, \ldots, x_*) \in \mathbb{R}^s$ is a strong attractor for the system associated with G in $(0, \beta)^s$.

Assuming excitatory connections, the previous theorems are optimal when f is a monotone function, satisfying the usual conditions of monotonicity ($f'(x) > 0$ for all x) and strong convexity ($xf''(x) < 0$ for all $x \neq 0$).

Assuming excitatory connections, the previous theorems are optimal when f is a monotone function, satisfying the usual conditions of monotonicity ($f'(x) > 0$ for all x) and strong convexity ($xf''(x) < 0$ for all $x \neq 0$).
Indeed, in this case, either 0 is the only fixed point of f, and then all solutions of (16) converge to zero,
or there are two nontrivial fixed points of f, $x_- < 0 < x_+$, and then $(x_+, \ldots, x_+)$ and $(x_-, \ldots, x_-)$ are equilibria of (16) attracting

$$\{\phi \in \mathcal{C}([-\tau, 0], \mathbb{R}^s) : \phi(t) \in (-\infty, 0)^s \textit{ for all } -\tau \leq t \leq 0\}$$

$$\{\phi \in \mathcal{C}([-\tau, 0], \mathbb{R}^s) : \phi(t) \in (0, \infty)^s \textit{ for all } -\tau \leq t \leq 0\}$$

respectively.

For more general activation functions, our result is also sharp in some architectures of the network, namely in the ring of an even number of neurons. Indeed, consider the system

$$x_i'(t) = -x_i(t) + f(x_{i+1}(t - \tau_i)), \; 1 \leq i \leq 2s \qquad (25)$$

(with the convention mod $(2s)$).

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$$x_i'(t) = -x_i(t) + f(x_{i+1}(t - \tau_i)),\ 1 \leq i \leq 2s \tag{25}$$

(with the convention mod $(2s)$).
Under mild conditions, if x_* is not an attractor of f in $(0, b)$, then there is a two cycle $\{y_1, y_2\}$ for f with $y_1 < x_* < y_2 = f(y_1)$. This 2-cycle produces two equilibria of (25) different from $(x_*, \ldots, x_*)$, namely, $(y_1, y_2, \ldots, y_1, y_2)$ and $(y_2, y_1, \ldots, y_2, y_1)$.

Consider

$$x_i'(t) = -x_i(t) + f(x_{i+1}(t - \tau_i)), \; 1 \leq i \leq 2s \tag{26}$$

Theorem

Assume that $y_1 \in \mathbb{R}$ is an attractor for

$$y(N+1) = f^2(y(N)) \tag{27}$$

in (a, b) and f is strictly decreasing in this interval. Then every solution of our system with initial condition in

$$\{\phi \in \mathcal{C}([-\tau, 0], \mathbb{R}^s) : \phi(t) \in ((a, b) \times f((a, b)))^s \;\; \textit{for all } -\tau \leq t \leq 0\}$$

satisfies that $\lim_{t\to\infty} x(t, \phi) = (y_1, f(y_1), \ldots, y_1, f(y_1))$.

Proof

Consider $T : \mathbb{R}^{2s} \longrightarrow \mathbb{R}^{2s}$ given by

$$T(x_1, \ldots, x_{2s}) = (f(x_2), f(x_3), \ldots, f(x_{2s}), f(x_1)).$$

Assume that $f : (a, b) \longrightarrow f((a, b)) \subset (a, b)$ is a continuous function so that x_* is an attractor for

$$x(N+1) = f^2(x(N)), \;\; N = 0, 1, \ldots, \tag{28}$$

in the interval (a, b) and f is strictly decreasing in this interval. Then

$$(x_*, f(x_*), x_*, f(x_*), \ldots, x_*, f(x_*)) \in \mathbb{R}^{2s}$$

is a strong attractor for the system associated with T in the set

$$((a, b) \times f((a, b)))^s \, .$$

BREAK TIME!

Global bifurcation results

In an easy way, the previous results can be used to provide criteria of global bifurcation. To illustrate this point of view, we consider system

$$x_i'(t) = -x_i(t) + f(x_{i+1}(t - \tau_i)),\ 1 \le i \le 2s \qquad (29)$$

with Morita's activation function

$$f(x) = \frac{1 - \exp(-\alpha x)}{1 + \exp(-\alpha x)} \times \frac{1 + k \exp(\beta(|x| - h))}{1 + \exp(\beta(|x| - h))}, \qquad (30)$$

where h, k, α, β are real parameters, $\alpha > 0$, $\beta > 0$.

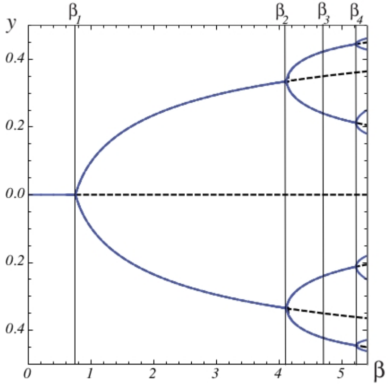
Next we describe analytically some numerical bifurcations studied in

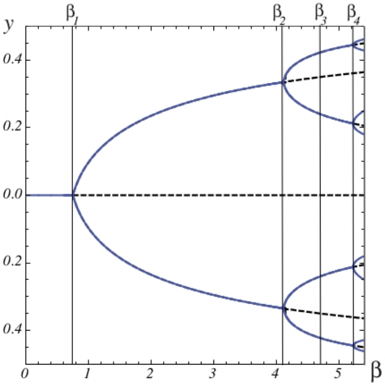
- J. Ma, J. Wu, Multistability and gluing bifurcation to butterflies in coupled networks with non-monotonic feedback, Nonlinearity 22 (2009) 1383–1412.

We consider the numerical example studied in that paper, that corresponds to $k = -0.8$, $h = 0.5$, and $\alpha = 7.5$, and use β as the bifurcation parameter. The bifurcation diagram of

$$x(N+1) = f_\beta(x(N))$$

is shown in the next figure.





Let $\beta_1 = 0.749387$ be the unique value of β for which $f'(0) = 1$. For $\beta \in (0, \beta_1)$, $f'(0) < 1$ and 0 is the unique solution of equation $f^2(x) = x$. Therefore 0 is a global attractor for the one dimensional equation in $\mathbb{R}$, and it follows from our theorems that $0 \in \mathbb{R}^{2s}$ is a global attractor of the system.

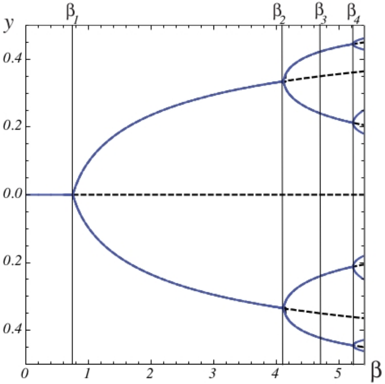
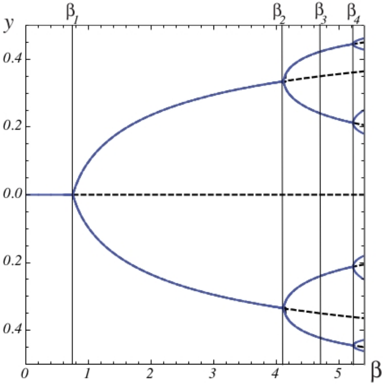


Figure: Bifurcation diagram of f for $k = -0.8$, $h = 0.5$, and $\alpha = 7.5$.

At $\beta = \beta_1$, there is a pitchfork bifurcation in the one dimensional equation producing two nontrivial fixed points $x_+ \in (0, x_0^+)$ and $x_- \in (x_0^-, 0)$, while 0 becomes unstable. Next, define $\beta_2 = 4.12413$ and for $\beta \in (\beta_1, \beta_2)$, there are exactly three fixed points of f (namely, $0, x_+, x_-$), and no other 2-periodic point.

Thus, x_+ attracts $(0, x_0^+)$, and our theorem ensures that $(x_+, \ldots, x_+)$ attracts all solutions of of the system with initial condition in the set

$$\{\phi \in \mathcal{C}([-\tau, 0], \mathbb{R}^{2s}) : \phi(t) \in (0, x_0^+)^{2s} \text{ for all } -\tau \leq t \leq 0\}.$$



At $\beta = \beta_2$, the fixed points x_+ and x_- of the one dimensional equation lose their stability in a period-doubling bifurcation, giving rise to a pair of cycles of period 2, which we denote by $\{y_1^+, y_2^+\}$ and $\{y_1^-, y_2^-\}$. They satisfy the inequalities

$$y_2^- < x_- < y_1^- < 0 < y_1^+ < x_+ < y_2^+.$$

For $\beta \in (\beta_2, \beta_3)$, let us consider the 2-periodic orbit $\{y_1^+, y_2^+\}$. The point y_1^+ is an attractor of f^2 in $(f^2(x_M^+), x_+)$, y_2^+ is an attractor of f^2 in $(x_+, f(x_M^+))$, and f is decreasing in both intervals. Then, our criterion applies to establish that the equilibrium $(y_1^+, y_2^+, \ldots, y_1^+, y_2^+)$ attracts all solutions of (25) with initial condition in

$$\{\phi \in \mathcal{C}([-\tau, 0], \mathbb{R}^{2s}) : \phi(t) \in (f^2(x_M^+), x_+)^s \times (x_+, f^3(x_M^+))^s$$

$$\text{for all } -\tau \leq t \leq 0\},$$

Applications in systems with dependence on time

Consider system

$$x_i'(t) = -x_i(t) + \sum_{j=1}^{s} a_{ij}(t) f_{ij}(x_j(t - \tau_{ij})), \; 1 \leq i \leq s. \qquad (31)$$

Theorem

Suppose that functions $f_{ij} : \mathbb{R} \longrightarrow \mathbb{R}$ satisfy conditions **(Ac1)–(Ac3)** *for all $i, j \in \{1, \dots, s\}$, and the coefficients $a_{ij} : \mathbb{R} \longrightarrow \mathbb{R}$ are continuous and bounded.*
If, for every $i, j \in \{1, \dots, s\}$, 0 is an attractor in $\mathbb{R}$ for the discrete equation

$$y(N+1) = \left(\sum_{k=1}^{s} |a_{ik}|_\infty \right) f_{ij}(y(N)), \; N = 0, 1, \dots, \qquad (32)$$

then all solutions $x(t)$ of (31) satisfy that $\lim_{t \to \infty} x_i(t) = 0$

The problem of global stability for neural systems of Hopfield type with variable coefficients has been addressed in several papers, usually assuming some property of recurrence on the coefficients $a_{ij}(t)$, such as periodicity or almost periodicity . Note that we do not impose any property of recurrence on the coefficients.

- X.-H. Tang, X Zou, The existence of global exponential stability of a periodic solutionof a class of delay differential equations, Nonlinearity 22 (2009) 2423–2442.
- S. Novo, R. Obaya, A. Sanz, Attractor minimal sets for non-autonomous delay functional differential equations with applications for neural networks, Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci. 461 (2005) 2767–2783.

Our strategy vs Lyapunov functions

$$x_i'(t) = -x_i(t) + \sum_{j=1}^{s} a_{ij} f_j(x_j(t-\tau_{ij})),\ 1 \leq i \leq s. \tag{33}$$

Assume that, for each $j \in \{1, 2, \ldots, n\}$, $f_j : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz with constant L_j and $f_{ij}(0) = 0$. It is easy to prove that if

$$\max_{1 \leq i \leq s} \left\{ \sum_{j=1}^{s} |a_{ij}| L_j \right\} < 1, \tag{34}$$

holds then every solution of our system converges to 0, regardless the value of the delays.

Similar conditions to (34) have been used in the literature for proving global attraction, using Lypaunov functions; see, e.g.,

- P. van den Driessche, X. Zou, Global attractivity in delayed Hopfield neural network models, SIAM J. Appl. Math. 58 (1998) 1878–1890.

Discrete models of Hopfield type

$$x_i(N+1) = \alpha x_i(N) + \sum_{j=1}^{n} a_{ij} f(x_j(N - k_{ij})),\ N = 0, 1, \ldots, \qquad (35)$$

for $i = 1, ..., n$, where $k_{ij} \in$, $a_{ij} \in \mathbb{R}$, for all $i, j \in \{1, 2, \ldots, n\}$.

We can obtain "essentially" the same results of attractivity as before. Now we have to consider the dynamics of the function $g(x) = (1/(1-\alpha))f(x)$. However, the strategy and the tools are completely different.

Coexistence of chaos and global attractivity

This problem is motivated by

- Y. Huang and X. Zou, Co-existence of chaos and stable periodic orbits in a simple discrete neural network , *J. Nonlinear Sci.* **15** (2005), 291–303.

Analytically, we are able to prove in our paper that if we have, for instance, an activation function with the graph illustrated on the blackboard, there are stable equilibria with large basins of attraction, together with regions with complex dynamics.

Conclusions

- We have deduced some properties of global attractivity and bifurcation in systems of delay differential equations from some dynamical behaviors of an equation in one dimension.

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THANK YOU VERY MUCH FOR YOUR ATTENTION